

Kernel Identities and Vectorial Regularization

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Abstract. We present the method of “vectorial regularization” to prove kernel identities. This method is applied to derive both known kernel identities, e.g. $\dot{\mathcal{B}}_{xy} = \dot{\mathcal{B}}_x \hat{\otimes}_\varepsilon \dot{\mathcal{B}}_y$, $\mathcal{D}'_{L^1,xy} = \mathcal{D}'_{L^1,x} \hat{\otimes}_\pi \mathcal{D}'_{L^1,y}$, as well as new ones: $\dot{\mathcal{B}}'_{xy} = \dot{\mathcal{B}}'_x \hat{\otimes}_\varepsilon \dot{\mathcal{B}}'_y$ and $\mathcal{D}_{L^1,xy} = \mathcal{D}_{L^1,x} \hat{\otimes}_\pi \mathcal{D}_{L^1,y}$.

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1 Introduction

In the following all distribution spaces are defined on the whole of \mathbb{R}^n , i.e., $\mathcal{D}' = \mathcal{D}'(\mathbb{R}^n)$, $\mathcal{D}'_{L^p} = \mathcal{D}'_{L^p}(\mathbb{R}^n)$, etc.

A *regularization property* for a distribution $T \in \mathcal{D}'$ is a statement of the following form: Let $1 \leq p \leq \infty$. For $T \in \mathcal{D}'$ the assertions

1. $T \in \mathcal{D}'_{L^p}$
2. $\forall \varphi \in \mathcal{D}: \varphi * T \in L^p$

are equivalent. By means of the “associated difference kernel”

$$T(x - y) \in \mathcal{D}'_{xy} = \mathcal{D}'(\mathbb{R}_x^n \times \mathbb{R}_y^n)$$

the equivalence above can be translated into the equivalence

$$T \in \mathcal{D}'_{L^p} \Leftrightarrow T(x - y) \in \mathcal{D}'_y \hat{\otimes} L^p_x.$$

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The “associated difference kernel” $T(x - y)$ is defined in [10, pp. 103–104].

A *vectorial regularization property* reads as: Let E be a space of distributions and $K(x, z) \in \mathcal{D}'_x(E_z)$. Then,

$$K(x, z) \in \mathcal{D}'_{L^p, x}(E_z) \Leftrightarrow K(x - y, z) \in (\mathcal{D}'_y \widehat{\otimes} L^p_x)(E_z).$$

The space $\mathcal{D}'_x(E_z)$ of E -valued distributions is defined as the subspace $\mathcal{D}'_x \varepsilon E_z$ of \mathcal{D}'_{xz} wherein the ε -product is defined in [10, p. 18]. Note that by Corollaire 1 in [10, p. 47] $\mathcal{D}' \varepsilon E = \mathcal{D}' \widehat{\otimes}_\varepsilon E$ if E is complete.

The symbol \otimes without subscript is used if $\otimes_\pi = \otimes_\varepsilon$, e.g., if one of the spaces is nuclear.

If $\mathcal{D}'_{L^p} \widehat{\otimes}_\pi E$ is used instead of $\mathcal{D}'_{L^p} \widehat{\otimes}_\varepsilon E$ we speak of vectorial regularization with the completed *projective* tensor product.

By *kernel identities*, we understand statements as e.g. L. Schwartz’s classical kernel theorem, i.e., $\mathcal{D}'_{xy} = \mathcal{D}'_x \widehat{\otimes} \mathcal{D}'_y$. Two fundamental examples of kernel identities are given in [5, chap. I, pp. 61, 90]:

$$L^1(X) \widehat{\otimes}_\pi L^1(Y) = L^1(X \times Y) \quad \text{and} \quad \mathcal{C}_0(X) \widehat{\otimes}_\varepsilon \mathcal{C}_0(Y) = \mathcal{C}_0(X \times Y), \quad (1)$$

where X and Y are locally compact spaces. In order to abbreviate the notation, we will write these identities for $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$ as

$$L^1_x \widehat{\otimes}_\pi L^1_y = L^1_{xy} \quad \text{and} \quad \mathcal{C}_{0,x} \widehat{\otimes}_\varepsilon \mathcal{C}_{0,y} = \mathcal{C}_{0,xy}.$$

In [10], L. Schwartz found the algebraic and topological kernel identities

$$\mathcal{D}'_{L^1, x} \widehat{\otimes}_\pi \mathcal{D}'_{L^1, y} = \mathcal{D}'_{L^1, xy} \quad (2)$$

for integrable distributions (Proposition 38 in [10, p. 135]) and

$$\dot{\mathcal{B}}_x \widehat{\otimes}_\varepsilon \dot{\mathcal{B}}_y = \dot{\mathcal{B}}_{xy} \quad (3)$$

for smooth functions with derivatives vanishing at infinity (Proposition 27 in [10, p. 59]). Further kernel identities are given in Proposition 28 in [10, p. 98],

$$\begin{aligned} \mathcal{E}_{xy} &= \mathcal{E}_x \widehat{\otimes} \mathcal{E}_y, & \mathcal{S}_{xy} &= \mathcal{S}_x \widehat{\otimes} \mathcal{S}_y, & \mathcal{O}_{M,xy} &= \mathcal{O}_{M,x} \widehat{\otimes} \mathcal{O}_{M,y}, \\ \mathcal{E}'_{xy} &= \mathcal{E}'_x \widehat{\otimes} \mathcal{E}'_y, & \mathcal{S}'_{xy} &= \mathcal{S}'_x \widehat{\otimes} \mathcal{S}'_y \end{aligned} \quad (4)$$

Note that however,

$$\mathcal{D}_{xy} = \mathcal{D}_x \widehat{\otimes}_\iota \mathcal{D}_y \quad \text{and} \quad \mathcal{O}_{C,xy} = \mathcal{O}_{C,x} \widehat{\otimes}_\iota \mathcal{O}_{C,y}$$

by Proposition 1 bis. in [11, p. 17] and [4, p. 34], respectively. Here \otimes_ι denotes the inductive tensor product defined in Definition 3 in [5, Chap. I, p. 74] and $\widehat{\otimes}_\iota$ its completion.

The aim of this article is to prove, in a uniform manner, known and new kernel identities, as (2) in Proposition 5, (3) in Proposition 2,

$$\mathcal{D}_{L^1, xy} = \mathcal{D}_{L^1, x} \widehat{\otimes}_\pi \mathcal{D}_{L^1, y} \quad (5)$$

in Proposition 6 and

$$\dot{\mathcal{B}}'_{xy} = \dot{\mathcal{B}}' \widehat{\otimes}_\varepsilon \dot{\mathcal{B}}'_y \quad (6)$$

in Proposition 3 by vectorial regularization properties. Our proofs of the identities (2, 3, 5, 6) show that they are all consequences of Grothendieck's fundamental examples (1).

Also it turns out that in some cases the topological part of the kernel identities follows from the algebraic identity and abstract structural results, e.g. for complete spaces of distributions \mathcal{H} the continuous embeddings

$$\mathcal{H}_x \widehat{\otimes}_\varepsilon \mathcal{H}_y \hookrightarrow \mathcal{D}'_x \widehat{\otimes}_\varepsilon \mathcal{D}'_y = \mathcal{D}'_{xy} \hookleftarrow \mathcal{H}_{xy}$$

imply that the identity mapping $\mathcal{H}_{xy} \rightarrow \mathcal{H}_x \widehat{\otimes}_\varepsilon \mathcal{H}_y$ has a closed graph. In concrete cases, sequence-space representations can be used to check whether these spaces satisfy the assumptions of a suitable closed graph theorem.

We use the notations of L. Schwartz in [12], e.g. the space of distributions \mathcal{E} , \mathcal{E}' , \mathcal{D} , \mathcal{D}' , \mathcal{D}_{L^p} , \mathcal{D}'_{L^p} for $1 \leq p \leq \infty$, $\dot{\mathcal{B}}$ and $\dot{\mathcal{B}}'$ (which is not the dual of $\dot{\mathcal{B}}$ but the closure of \mathcal{E}' in \mathcal{D}'_{L^∞}). For *vector-valued distributions*, constant use is made of L. Schwartz' treatise [10, 11]. Instead of $K(\hat{x}, \hat{y})$ we simply write $K(x, y)$ for kernels $K(x, y) \in \mathcal{D}'_{xy}$.

Proposition 3 was presented in a talk given by the second author in Vienna, June, 2015.

2 Regularization and the injective tensor product

Proposition 1. *Let E be a complete space of distributions. For a kernel $K(x, z) \in \mathcal{D}'_{xz}$ the following characterizations hold:*

1. $K(x, z) \in \dot{\mathcal{B}}'_x \widehat{\otimes}_\varepsilon E_z \Leftrightarrow K(x - y, z) \in (\mathcal{D}'_y \widehat{\otimes} \mathcal{C}_{0,x}) \widehat{\otimes}_\varepsilon E_z$.
2. $K(x, z) \in \dot{\mathcal{B}}_x \widehat{\otimes}_\varepsilon E_z \Leftrightarrow K(x - y, z) \in (\mathcal{E}_y \widehat{\otimes} \mathcal{C}_{0,x}) \widehat{\otimes}_\varepsilon E_z$.

Although for a normal space of distributions E the characterization 1 of this Proposition is a special case of Proposition 15 in [3], we include it nevertheless to keep the article self-contained.

Proof. 1. We first show the case of distributions.

\Rightarrow : The mapping

$$\tau: \dot{\mathcal{B}}' \rightarrow \mathcal{D}'_y \widehat{\otimes} \mathcal{C}_{0,x}, S \mapsto S(x - y)$$

is well-defined, linear and continuous according to Remarque 3 in [12, p. 202].

Hence also the mapping

$$\tau \varepsilon \text{id}_E: \dot{\mathcal{B}}'_x \widehat{\otimes}_\varepsilon E_z \rightarrow (\mathcal{D}'_y \widehat{\otimes} \mathcal{C}_{0,x}) \widehat{\otimes}_\varepsilon E_z, K(x, z) \mapsto K(x - y, z)$$

as by Proposition 1 in [10, p. 20] the ε -product of continuous linear mappings is again continuous.

\Leftarrow : Multiplication of $K(x - y, z) \in \mathcal{D}'_y(\mathcal{C}_{0,x} \widehat{\otimes}_\varepsilon E_z)$ with $\delta(w - y) \in \mathcal{D}_y \widehat{\otimes} \mathcal{D}'_w$ using Proposition 25 in [10, p. 120] leads to

$$\delta(w - y)K(x - y, z) \in \mathcal{E}'_y(\mathcal{D}'_w \widehat{\otimes} (\dot{\mathcal{B}}_x \widehat{\otimes}_\varepsilon E_z)) = (\mathcal{E}'_y \widehat{\otimes}_\varepsilon \mathcal{C}_{0,x}) \widehat{\otimes}_\varepsilon (\mathcal{D}'_w \widehat{\otimes}_\varepsilon E_z).$$

From $\mathcal{C}_{0,x} \widehat{\otimes}_\varepsilon \mathcal{E}'_y \hookrightarrow \dot{\mathcal{B}}'_{xy}$ and the invariance of $\dot{\mathcal{B}}'_{x,y}$ under the coordinate transform

$$\begin{aligned} x - y &= u & x &= u + v \\ y &= v & y &= v \end{aligned}$$

we deduce

$$\delta(v - w)K(u, z) \in \dot{\mathcal{B}}'_{uv} \widehat{\otimes}_\varepsilon \mathcal{D}'_w \widehat{\otimes}_\varepsilon E_z \subset (\mathcal{S}'_v \widehat{\otimes} \dot{\mathcal{B}}'_u)(\mathcal{D}'_w \widehat{\otimes}_\varepsilon E_z).$$

Evaluation with $e^{-|v|^2} \in \mathcal{S}_v$ yields

$$e^{-|w|^2}K(u, z) \in \dot{\mathcal{B}}'_u \widehat{\otimes}_\varepsilon \mathcal{D}'_w \widehat{\otimes}_\varepsilon E_z = \mathcal{D}'_w(\dot{\mathcal{B}}'_u \widehat{\otimes}_\varepsilon E_z).$$

Multiplication by $e^{|w|^2} \in \mathcal{E}_w$, which is possible by Theorem 7.1 in [9] leads to

$$K(u, z) \in \mathcal{D}'_w(\dot{\mathcal{B}}'_u \widehat{\otimes}_\varepsilon E_z)$$

and hence

$$K(u, z) \in \dot{\mathcal{B}}'_u \widehat{\otimes}_\varepsilon E_z.$$

2. The implication “ \Rightarrow ” is completely analogous to the case of distributions if we use that the convolution mapping $\ast: \mathcal{E}' \times \dot{\mathcal{B}} \rightarrow \mathcal{C}_0$ is well defined and hypocontinuous since $\mathcal{E}' \hookrightarrow \mathcal{D}'_{L^1}$ and $\dot{\mathcal{B}} \hookrightarrow \mathcal{C}_0$ (see e.g. [7]).
Let us show the implication “ \Leftarrow ”. The vectorial scalar product of

$$K(x - y, z) \in \mathcal{E}_y(\mathcal{C}_{0,x} \widehat{\otimes}_\varepsilon E_z)$$

with $\partial^\alpha \delta(y) \in \mathcal{E}'_y$ yields $\partial_x^\alpha K(x, z) \in \mathcal{C}_{0,x} \widehat{\otimes}_\varepsilon E_z$ for all $\alpha \in \mathbb{N}_0^n$. From this we deduce $K(x, z) \in \dot{\mathcal{B}}_x \widehat{\otimes}_\varepsilon E_z$ using the compatibility of the vector-valued scalar product with continuous linear mappings by [11, p. 18]. □

Remark 1. Note that it is possible to generalize this result to non-complete spaces of distributions but in this case the completed ε -tensor product has to be replaced by the ε -product.

Proposition 2 (see Proposition 17 in [10]). *The space of smooth functions vanishing at infinity satisfies the kernel identity*

$$\dot{\mathcal{B}}_{xy} = \dot{\mathcal{B}}_x \widehat{\otimes}_\varepsilon \dot{\mathcal{B}}_y$$

algebraically and topologically.

Proof. In order to show the algebraic part, observe that for $K(x, y) \in \mathcal{D}'_{xy}$ we get

$$\begin{aligned} K(x, y) \in \dot{\mathcal{B}}_{xy} &\Leftrightarrow K(x - z, y - w) \in \mathcal{E}_{zw} \widehat{\otimes}_\varepsilon \mathcal{C}_{0,xy} = (\mathcal{E}_z \widehat{\otimes} \mathcal{C}_{0,x}) \widehat{\otimes}_\varepsilon (\mathcal{E}_w \widehat{\otimes} \mathcal{C}_{0,y}) \\ &\Leftrightarrow K(x, y - w) \in \dot{\mathcal{B}}_x (\mathcal{E}_w \widehat{\otimes} \mathcal{C}_{0,y}) \\ &\Leftrightarrow K(x, y) \in \dot{\mathcal{B}}_x \widehat{\otimes}_\varepsilon \dot{\mathcal{B}}_y. \end{aligned}$$

Let us now show the topological identity as well. As the ε -product of two continuous linear mappings is again continuous, we see that the mapping

$$\dot{\mathcal{B}}_x \widehat{\otimes}_\varepsilon \dot{\mathcal{B}}_y \rightarrow \mathcal{C}_{0,x} \widehat{\otimes}_\varepsilon \mathcal{C}_{0,y} = \mathcal{C}_{0,xy}, f \mapsto \partial_x^\alpha \partial_y^\beta f$$

is continuous for all multi-indices α and β . Therefore the topology of $\dot{\mathcal{B}}_x \widehat{\otimes}_\varepsilon \dot{\mathcal{B}}_y$ is finer than the one of $\dot{\mathcal{B}}_{xy}$. Therefore these topologies are comparable Fréchet space topologies on the same vector space and hence they coincide. \square

Proposition 3. *The space of distributions vanishing at infinity satisfies the kernel identity*

$$\dot{\mathcal{B}}'_{xy} = \dot{\mathcal{B}}'_x \widehat{\otimes}_\varepsilon \dot{\mathcal{B}}'_y$$

algebraically and topologically.

Proof. For $K(x, y) \in \mathcal{D}'_{xy}$ we start with the characterization

$$K(x, y) \in \dot{\mathcal{B}}'_{xy} \Leftrightarrow K(x - z, y - w) \in \mathcal{D}'_{zw} \widehat{\otimes} \dot{\mathcal{B}}_{xy}$$

of $\dot{\mathcal{B}}'$ by regularization. From this we deduce

$$\begin{aligned} K(x, y) \in \dot{\mathcal{B}}'_{xy} &\Leftrightarrow K(x - z, y - w) \in (\mathcal{D}'_z \widehat{\otimes} \mathcal{D}'_w) \widehat{\otimes} (\dot{\mathcal{B}}_x \widehat{\otimes}_\varepsilon \dot{\mathcal{B}}_y) \\ &\Leftrightarrow K(x - z, y - w) \in (\mathcal{D}'_z \widehat{\otimes} \dot{\mathcal{B}}_x) \widehat{\otimes}_\varepsilon (\mathcal{D}'_w \widehat{\otimes} \dot{\mathcal{B}}_y). \end{aligned}$$

using the kernel theorem for $\dot{\mathcal{B}}$ and \mathcal{D}' as well as the commutativity of the ε -tensor product. From this we get from Proposition 1,

$$\begin{aligned} K(x, y) \in \dot{\mathcal{B}}'_{xy} &\Leftrightarrow K(x, y - w) \in \dot{\mathcal{B}}'_x \widehat{\otimes}_\varepsilon (\mathcal{D}'_w \widehat{\otimes} \dot{\mathcal{B}}_y) = (\mathcal{D}'_w \widehat{\otimes} \dot{\mathcal{B}}_y) \widehat{\otimes}_\varepsilon \dot{\mathcal{B}}'_x \\ &\Leftrightarrow K(x, y) \in \dot{\mathcal{B}}'_y \widehat{\otimes}_\varepsilon \dot{\mathcal{B}}'_x, \end{aligned}$$

which proves the algebraic part of the kernel identity. Using the sequence space representation $\dot{\mathcal{B}}' = s' \widehat{\otimes}_\varepsilon c_0$ given in Theorem 3 in [2, p. 13] and

$$(s' \widehat{\otimes}_\varepsilon c_0) \widehat{\otimes}_\varepsilon (s' \widehat{\otimes}_\varepsilon c_0) \cong (s' \widehat{\otimes} s') \widehat{\otimes} (c_0 \widehat{\otimes}_\varepsilon c_0) \cong s' \widehat{\otimes}_\varepsilon c_0$$

we see by Proposition 7 in [2, p. 13] that both $\dot{\mathcal{B}}'_{xy}$ and $\dot{\mathcal{B}}'_x \widehat{\otimes}_\varepsilon \dot{\mathcal{B}}'_y$ are complete ultrabornological (DF)-spaces. From the continuity of the embeddings

$$\dot{\mathcal{B}}'_x \widehat{\otimes}_\varepsilon \dot{\mathcal{B}}'_y \hookrightarrow \mathcal{D}'_x \widehat{\otimes} \mathcal{D}'_y = \mathcal{D}'_{xy} \hookrightarrow \dot{\mathcal{B}}'_{xy},$$

we deduce that the identity mapping $\dot{\mathcal{B}}'_x \widehat{\otimes}_\varepsilon \dot{\mathcal{B}}'_y \rightarrow \dot{\mathcal{B}}'_{xy}$ has a closed graph. Therefore the topological identity follows by de Wilde's closed graph theorem (Theorem 5.4.1 in [6, p. 92]) since complete (DF)-spaces have a completing web by Proposition 12.4.6 in [6, p. 260]. \square

3 Regularization and the projective tensor product

In order to proof a version of Proposition 1 for the projective tensor product, we need the following lemma.

Lemma 1. *For $1 < q < \infty$ the following continuous embeddings hold:*

$$\mathcal{S}_x \widehat{\otimes} \mathcal{D}_{L^q, y} \hookrightarrow \mathcal{D}_{L^q, xy} \hookrightarrow \mathcal{E}_x \widehat{\otimes} \mathcal{D}_{L^q, y},$$

i.e., these spaces are contained with a finer topology. Moreover these spaces are contained as dense subspaces.

Proof. From $\mathcal{E}_{xy} = \mathcal{E}_x \widehat{\otimes} \mathcal{E}_y$, we deduce that $\mathcal{D}_{L^q, x} \widehat{\otimes} \mathcal{E}_y$ is a space of smooth functions. Using Lebesgue's theorem on dominated convergence we conclude that for $f \in \mathcal{D}_{L^q, xy}$ the function $\mathbb{R}_x^d \rightarrow \mathcal{D}_{L^q, y}, x \mapsto f(x, \cdot)$ has continuous derivatives of all order. Continuity of the embedding $\mathcal{D}_{L^q, xy} \hookrightarrow \mathcal{E}_x \widehat{\otimes} \mathcal{D}_{L^q, y}$ follows inductively from the Sobolev trace theorem, see, e.g., Theorem 5.36 in [1].

Given $f \in \mathcal{S}_x \widehat{\otimes} \mathcal{D}_{L^q, y}$, the inequality

$$\begin{aligned} \int_{\mathbb{R}^{d_1+d_2}} |f(x, y)|^p \, dx \, dy &= \int_{\mathbb{R}^{d_1+d_2}} (1 + |x|^2)^{-d_1-1} (1 + |x|^2)^{d_1+1} |f(x, y)|^p \, dx \, dy \\ &\leq \int_{\mathbb{R}^{d_1}} (1 + |x|^2)^{-d_1-1} \, dx \sup_{x \in \mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} (1 + |x|^2)^{d_1+1} |f(x, y)|^p \, dy \\ &\leq C \sup_{x \in \mathbb{R}^{d_2}} (1 + |x|^2)^{d_1+1} \int_{\mathbb{R}^{d_2}} |f(x, y)|^p \, dy \end{aligned}$$

proves $\mathcal{S}_x \widehat{\otimes} \mathcal{D}_{L^q, y} \hookrightarrow \mathcal{D}_{L^q, xy}$. The spaces are contained as dense subspaces since

$$\mathcal{D}_{xy} \hookrightarrow \mathcal{D}_x \widehat{\otimes} \mathcal{D}_y \subset \mathcal{S}_x \widehat{\otimes} \mathcal{D}_{L^q, y}$$

and the injective tensor product preserves dense subspaces by Proposition 16.2.5 in [6, p. 349]. \square

Proposition 4. *Let E be a space of distributions and $1 \leq p < \infty$. For $K(x, z) \in \mathcal{D}'_{xz}$ the following characterizations hold:*

1. $K(x, z) \in \mathcal{D}'_{L^p, x} \widehat{\otimes}_\pi E_z \Leftrightarrow K(x - y, z) \in (\mathcal{D}'_y \widehat{\otimes} L_x^p) \widehat{\otimes}_\pi E_z.$
2. $K(x, z) \in \mathcal{D}_{L^p, x} \widehat{\otimes}_\pi E_z \Leftrightarrow K(x - y, z) \in (\mathcal{E}_y \widehat{\otimes} L_x^p) \widehat{\otimes}_\pi E_z.$

Proof. 1. \Rightarrow : The mapping $\tau: \mathcal{D}'_{L^p} \rightarrow \mathcal{D}'_y \widehat{\otimes} L_x^p, S \mapsto S(x - y)$ is well-defined, linear and continuous according to [12, p. 204]. Hence also

$$\tau \otimes \text{id}_E: \mathcal{D}'_{L^p, x} \widehat{\otimes}_\pi E_z \rightarrow (\mathcal{D}'_y \widehat{\otimes} L_x^p) \widehat{\otimes}_\pi E_z, K(x, z) \mapsto K(x - y, z)$$

as the π -tensor product of continuous linear mappings is again a continuous and linear mapping.

\Leftarrow : Multiplication of $K(x-y, z) \in \mathcal{D}'_y(L^p_x \widehat{\otimes}_\pi E_z)$ with $\delta(w-y) \in \mathcal{D}_y \widehat{\otimes} \mathcal{D}'_w$ according to Proposition 25 in [11, p. 120] yields

$$\begin{aligned} \delta(w-y)K(x-y, z) &\in \mathcal{E}'_y(L^p_x \widehat{\otimes}_\pi (\mathcal{D}'_w \widehat{\otimes}_\pi E_z)) = \mathcal{E}'_y \widehat{\otimes}_\pi L^p_x \widehat{\otimes}_\pi \mathcal{D}'_w \widehat{\otimes}_\pi E_z \\ &= \mathcal{D}'_w(\mathcal{E}'_y(L^p_x)) \widehat{\otimes}_\pi E_z \\ &\subset (\mathcal{D}'_w \widehat{\otimes} \mathcal{D}'_{L^p, xy}) \widehat{\otimes}_\pi E_z. \end{aligned}$$

Note that the inclusion $L^p \widehat{\otimes} \mathcal{E}' \subset \mathcal{D}'_{L^p}$ follows from $L^p \widehat{\otimes} \mathcal{E}' \subset \mathcal{D}'_{L^p} \widehat{\otimes} \mathcal{E}'$ and from $\dot{\mathcal{B}}_{xy} = \dot{\mathcal{B}}_x \widehat{\otimes}_\varepsilon \dot{\mathcal{B}}_y \hookrightarrow \dot{\mathcal{B}}_x \widehat{\otimes} \mathcal{E}'_y$ for $p = 1$ and from Lemma 1 and $(\dot{\mathcal{B}} \widehat{\otimes} \mathcal{E})' = \mathcal{D}'_{L^1_x} \widehat{\otimes} \mathcal{E}'_y$ and $(\mathcal{D}_{L^q} \widehat{\otimes} \mathcal{E})' = \mathcal{D}'_{L^p} \widehat{\otimes} \mathcal{E}'$ by Théorème 12 in [5, chap. II, p. 76] for $1 < p < \infty$. Using the coordinate transform

$$\begin{array}{ll} x-y=u & x=u+v \\ y=v & y=v \end{array}$$

we obtain

$$\delta(w-v)K(u, z) \in \mathcal{D}'_{L^p, u, v} \widehat{\otimes}_\pi (\mathcal{D}'_w \widehat{\otimes}_\pi E_z)$$

from the invariance of $\mathcal{D}'_{L^p, x, y}$ under coordinate transforms.

From $\mathcal{D}'_{L^p, x, y} \subset \mathcal{D}'_{L^p, x} \widehat{\otimes} \mathcal{S}'_y$ we deduce that the application of

$$\delta(w-v)K(u, z) \in \mathcal{S}'_v(\mathcal{D}'_w \widehat{\otimes} (\mathcal{D}'_{L^p, u} \widehat{\otimes}_\pi E_z))$$

to $e^{-|v|^2} \in \mathcal{S}_y$ is

$$e^{-|w|^2}K(u, z) \in \mathcal{D}'_w \widehat{\otimes} (\mathcal{D}'_{L^p, u} \widehat{\otimes}_\pi E_z).$$

Multiplication by $e^{|w|^2} \in \mathcal{E}_w$ according to Theorem 7.1 in [9, p. 31] yields $K(u, z) \in \mathcal{D}'_w \widehat{\otimes} (\mathcal{D}'_{L^p, u} \widehat{\otimes}_\pi E_z)$ and hence $K(u, z) \in \mathcal{D}'_{L^p, u} \widehat{\otimes}_\pi E_z$.

2. The implication “ \Rightarrow ” is completely analogous to the case of distributions if we use that the convolution mapping $\ast: \mathcal{E}' \times \mathcal{D}_{L^p} \rightarrow L^p$ is well defined and hypocontinuous since $\mathcal{E}' \hookrightarrow \mathcal{D}'_{L^1}$ and $\mathcal{D}_{L^p} \hookrightarrow L^p$ (see e.g. [7]).

Let us show the implication “ \Leftarrow ”. The vectorial scalar product of

$$K(x-y, z) \in \mathcal{E}_y \widehat{\otimes} (L^p_x \widehat{\otimes}_\pi E_z)$$

with $\partial^\alpha \delta(y) \in \mathcal{E}'_y$ yields $\partial^\alpha_x K(x, z) \in L^p_x \widehat{\otimes}_\pi E_z$ for all $\alpha \in \mathbb{N}_0^n$. From this we deduce $K(x, z) \in \mathcal{D}_{L^p, x} \widehat{\otimes}_\pi E_z$ using the compatibility of the vector-valued scalar product with continuous linear mappings by [11, p. 18]. □

Remark 2. More general, the proof of equivalence 1 in Proposition 4 also works in the following situation. Let \mathcal{H}' be a space of distributions and \mathcal{K} a space of functions such that the convolution mapping $\mathcal{H}' \times \mathcal{D} \rightarrow \mathcal{K}$ is hypocontinuous. If additionally the embeddings

$$\mathcal{K}_x \widehat{\otimes} \mathcal{E}'_y \hookrightarrow \mathcal{H}'_{x, y} \hookrightarrow \mathcal{H}'_x \widehat{\otimes} \mathcal{S}'_y \tag{7}$$

are well-defined and continuous, for kernels $K(x, y) \in \mathcal{D}'_{x,y}$ we get the following equivalence

$$K(x, z) \in \mathcal{H}'_x \widehat{\otimes}_\pi E_z \Leftrightarrow K(x - y, z) \in (\mathcal{D}'_y \widehat{\otimes} \mathcal{K}_x) \widehat{\otimes}_\pi E_z.$$

Examples of spaces \mathcal{H}' satisfying condition (7) are duals of normal spaces of distributions \mathcal{H} where the embeddings

$$\mathcal{S}_x \widehat{\otimes} \mathcal{H}_y \hookrightarrow \mathcal{H}_{x,y} \hookrightarrow \mathcal{E}_x \widehat{\otimes} \mathcal{H}_y$$

are well-defined and continuous. Note that the spaces $\mathcal{S} \widehat{\otimes} \mathcal{H}$ and $\mathcal{E} \widehat{\otimes} \mathcal{H}$ are spaces of \mathcal{H} -valued smooth functions. We refer to [8] for a detailed treatment of these spaces.

In the following we will discuss two kernel-identities as applications of Proposition 4.

Proposition 5 (see Proposition 38 in [10, p. 135]). *The space of integrable distributions satisfies the kernel identity*

$$\mathcal{D}'_{L^1, x, y} = \mathcal{D}'_{L^1, x} \widehat{\otimes}_\pi \mathcal{D}'_{L^1, y}$$

algebraically and topologically.

Proof. For $K(x, y) \in \mathcal{D}'_{x,y}$ we have the equivalence

$$K(x, y) \in \mathcal{D}'_{L^1, x, y} \Leftrightarrow K(x - z, y - w) \in \mathcal{D}'_{z,w} \widehat{\otimes} L^1_{x,y}$$

which follows from the characterization of \mathcal{D}'_{L^1} by regularization given in Théorème XXV in [12, p. 201]. Using the kernel identities $\mathcal{D}'_{x,y} = \mathcal{D}'_x \widehat{\otimes} \mathcal{D}'_y$ and $L^1_{xy} = L^1_x \widehat{\otimes}_\pi L^1_y$, we obtain

$$K(x - z, y - w) \in \mathcal{D}'_z \widehat{\otimes} \mathcal{D}'_w (L^1_x \widehat{\otimes}_\pi L^1_y) = (\mathcal{D}'_z \widehat{\otimes} L^1_x) \widehat{\otimes}_\pi (\mathcal{D}'_w \widehat{\otimes}_\pi L^1_y).$$

Applying Proposition 4 twice to the line above, we finally get

$$\begin{aligned} K(x, y) \in \mathcal{D}'_{L^1, x, y} &\Leftrightarrow K(x - z, y - w) \in (\mathcal{D}'_z \widehat{\otimes} L^1_x) \widehat{\otimes}_\pi (\mathcal{D}'_w \widehat{\otimes}_\pi L^1_y) \\ &\Leftrightarrow K(x, y) \in \mathcal{D}'_{L^1, x} \widehat{\otimes}_\pi \mathcal{D}'_{L^1, y}, \end{aligned}$$

i.e. we have shown the algebraic identity $\mathcal{D}'_{L^1, xy} = \mathcal{D}'_{L^1, x} \widehat{\otimes}_\pi \mathcal{D}'_{L^1, y}$.

In order to prove the continuity of the identity mapping

$$\mathcal{D}'_{L^1, x} \widehat{\otimes}_\pi \mathcal{D}'_{L^1, y} \rightarrow \mathcal{D}'_{L^1, xy}$$

it is sufficient to show the continuity of the bilinear mapping

$$\mathcal{D}'_{L^1, x} \times \mathcal{D}'_{L^1, y} \rightarrow \mathcal{D}'_{L^1, xy}, (S(x), T(y)) \mapsto S(x) \otimes T(y).$$

The continuity of this mapping follows from the separate continuity due to the fact that for (DF)-spaces separate continuity of bilinear maps implies continuity. The separate continuity follows immediately by the closed graph theorem.

By de Wilde's closed graph theorem (Theorem 5.4.1 in [6, p. 92]) the identity is a topological isomorphism because $\mathcal{D}'_{L^1, xy}$ is ultrabornological and $\mathcal{D}'_{L^1, x} \widehat{\otimes}_\pi \mathcal{D}'_{L^1, y}$ is a complete (DF)-space and, hence, has a completing web by Proposition 12.4.6 in [6, p. 260]. \square

Proposition 6. *The space of integrable smooth functions satisfies the kernel identity*

$$\mathcal{D}_{L^1,xy} = \mathcal{D}_{L^1,x} \widehat{\otimes}_\pi \mathcal{D}_{L^1,y}$$

algebraically and topologically.

Proof. For $S \in \mathcal{D}'$ we get

$$S \in \mathcal{D}_{L^1} \Leftrightarrow S(x-y) \in \mathcal{E}_y \widehat{\otimes} L_x^1$$

and therefore for $K \in \mathcal{D}'_{xy}$,

$$K(x,y) \in \mathcal{D}_{L^1,xy} \Leftrightarrow K(x-z, y-w) \in \mathcal{E}_{zw} \widehat{\otimes} L_{xy}^1.$$

From this equivalence, we deduce

$$K(x,y) \in \mathcal{D}_{L^1,xy} \Leftrightarrow K(x-z, y-w) \in \mathcal{E}_z \widehat{\otimes} \mathcal{E}_w \widehat{\otimes} (L_x^1 \widehat{\otimes}_\pi L_y^1) = (\mathcal{E}_z \widehat{\otimes} L_x^1) \widehat{\otimes}_\pi (\mathcal{E}_w \widehat{\otimes} L_y^1)$$

using the classical kernel identities $\mathcal{E}_{xy} = \mathcal{E}_x \widehat{\otimes} \mathcal{E}_y$ and $L_{xy}^1 = L_x^1 \widehat{\otimes}_\pi L_y^1$. From Proposition 4, applied twice, we get

$$K(x,y) \in \mathcal{D}_{L^1,xy} \Leftrightarrow K(x, y-w) \in \mathcal{D}_{L^1,x} \widehat{\otimes}_\pi (\mathcal{E}_w \widehat{\otimes} L_y^1) \Leftrightarrow K(x,y) \in \mathcal{D}_{L^1,xy}.$$

As the π -tensor product of continuous mappings is continuous, the mapping

$$\mathcal{D}_{L^1,x} \widehat{\otimes}_\pi \mathcal{D}_{L^1,y} \rightarrow L_x^1 \widehat{\otimes}_\pi L_y^1 = L_{xy}^1, \quad f \mapsto \partial_x^\alpha \partial_y^\beta f$$

is continuous for all multi-indices α and β . Hence the π -topology is finer than the topology of \mathcal{D}_{L^1} . As these topologies are comparable Fréchet space topologies on the same vector space they coincide by the closed graph theorem. \square

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